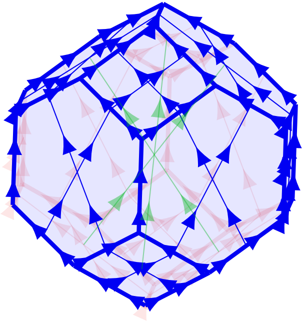
Aram Dermenjian (York Uni), Christophe Hohlweg (LaCIM), Thomas McConville (UNC), Vincent Pilaud (LIX)

Welcome!

Thanks for coming to my poster talk! You can either go through the slides like "normal", or jump around using the links in green (ex: Go to directory) or in the bottom-right corner of every slide. If you have any questions, don't hesitate to ask Aram!



▸ Start with the directory

▸ Start with the main result!

Link to directory

arXiv: 1910.03511

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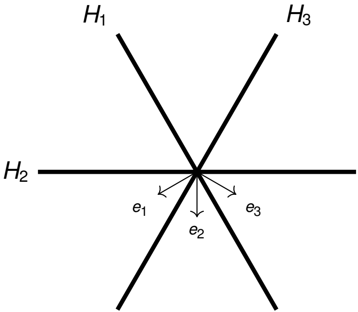
Come back at any time

Hyperplane arrangements

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dim real Euclidean vector space.

- A *hyperplane H* is $\operatorname{codim}\ 1$ subspace of V with normal e_H.
- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \dots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- Central $\mathcal{A}$ is *essential* if $\{0\} = \bigcap \mathcal{A}$.



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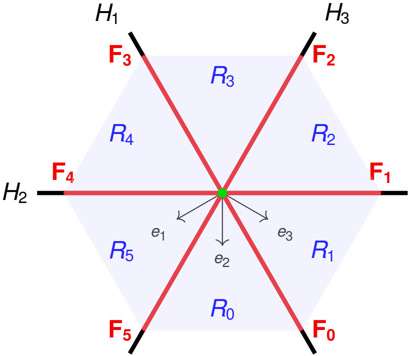
Regions and faces

Let $\mathcal{A}$ be an arrangement.

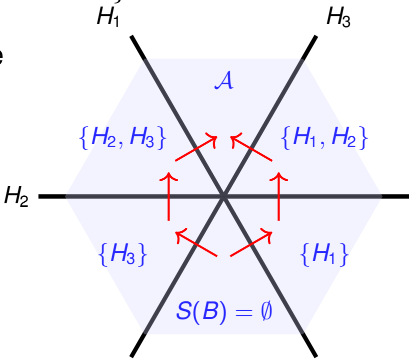
- *Regions* $\mathscr{R}_{\mathcal{A}}$ - connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_{\mathcal{A}}$ - intersections of closures of some regions.



Poset of regions

- *Base region B* - some fixed region in $\mathscr{R}_{\mathcal{A}}$.
- *Separation set for $R \in \mathscr{R}_{\mathcal{A}}$*
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$

The *poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ is the set of regions ordered by inclusion:
$R \leq_{\mathrm{PR}} R' \Leftrightarrow S(R) \subseteq S(R')$



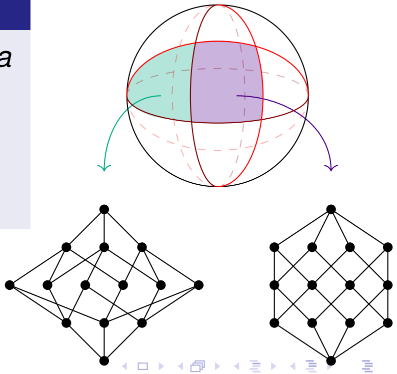
Lattice of regions

An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *simplicial* if every region is simplicial (*i.e.*, has n boundary hyperplanes).

Theorem (Björner, Edelman, Ziegler '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for any $B \in \mathscr{R}_{\mathcal{A}}$.

If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then B is simplicial.



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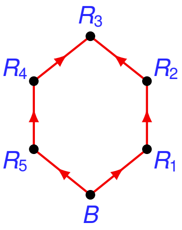
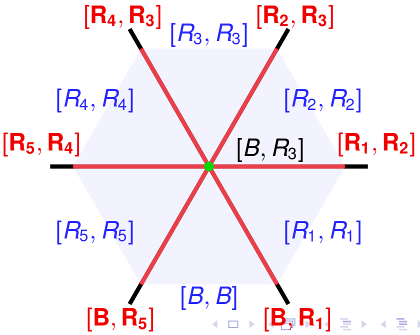
Facial intervals

Proposition (Björner, Las Vergnas, Sturmfels, White, Ziegler '93)

For every $F \in \mathscr{F}_\mathcal{A}$ there is a unique interval in $\mathrm{PR}(\mathcal{A}, B)$*:*

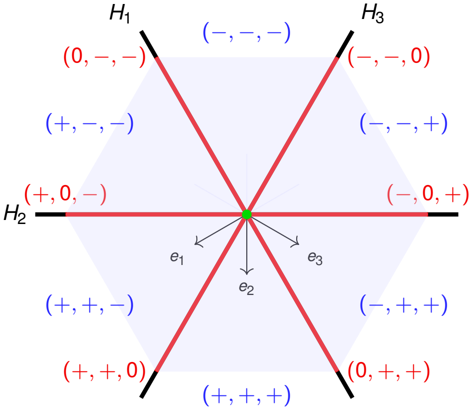
$$[m_F, M_F] = \left\{ R \in \mathscr{R}_\mathcal{A} \mid F \subseteq \overline{R} \right\}$$





Covectors

A *covector* of a face is a sign vector in $\{-, 0, +\}^{\mathcal{A}}$ relative to hyperplanes.



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Facial weak order

Let $\mathrm{PR}(\mathcal{A}, B)$ be the poset of regions, $[m_F, M_F]$ be the facial interval of a face F and $\mathcal{L}$ be the set of covectors.

The *facial weak order*, $\mathrm{FW}(\mathcal{A}, B)$, is the partial order $\leq_{\mathrm{FW}}$ on the set of faces (the left-hand definition). Let F, G by faces in $\mathscr{F}_{\mathcal{A}}$:

Definition

$$F \leq_{\mathrm{FW}} G$$
$$\Leftrightarrow$$
$$m_F \leq_{\mathrm{PR}} m_G$$
$$M_F \leq_{\mathrm{PR}} M_G$$

Definition

If $|\dim(F) - \dim(G)| = 1$ and

1. $F \subseteq G$, $M_F = M_G$, or
2. $G \subseteq F$, $m_F = m_G$.

then $F \lessdot G$.

Definition

$$F \leq_{\mathcal{L}} G$$
$$\Leftrightarrow$$
$$F(H) \geq G(H)$$
$$(\forall H \in \mathcal{A})$$

Theorem (Dermenjian, Hohlweg, McConville, Pilaud '19+)

$$(F \leq_{\mathrm{FW}} G) \quad \Leftrightarrow \quad (F = F_1 \lessdot \ldots \lessdot F_n = G) \quad \Leftrightarrow \quad (F \leq_{\mathcal{L}} G)$$

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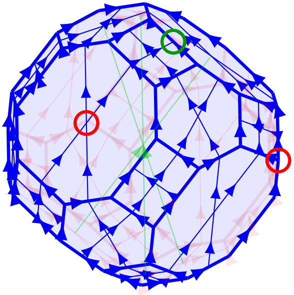
Main results

Theorem (Dermenjian, Hohlweg, McConville, Pilaud '19+)

Let $\mathcal{A}$ be an arrangement and fix a base region B. If the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then the facial weak order $\mathrm{FW}(\mathcal{A}, B)$ is a lattice.

B_3 Example:



Properties of the facial weak order
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Properties of the facial weak order

1. *Dual* of a poset P is the poset P^{op} where $x \leq_P y$ iff $y \leq_{P^{op}} x$. *Self-dual* if $P \cong P^{op}$.
2. A lattice is *semi-distributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$ and similarly for meets.
3. $x \in P$ is *join-irreducible* if it covers exactly one element.

Theorem (Dermenjian, Hohlweg, McConville, Pilaud '19+)

- *Facial weak order is self-dual.*
- *If $\mathcal{A}$ is simplicial then the facial weak order is semi-distributive.*
- *If $\mathcal{A}$ is simplicial then F is join-irreducible if and only if M_F is join-irreducible in* $\mathrm{PR}(\mathcal{A}, B)$ *and* $\mathrm{codim}(F) \in \{0, 1\}$.

The Möbius function for $X \leq Y$ is given by:

$$\mu(X, Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \text{if } X \leq Z \leq Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise} \end{cases}$$