The facial weak order in hyperplane arrangements

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[3]École Polytechnique (LIX)

30 May 2019

On this day in 1814 Eugène Catalan was born.

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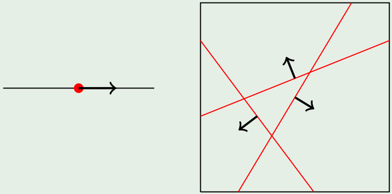
History and Background - Hyperplanes

- $(V, \langle \cdot, \cdot \rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is $\operatorname{codim}\ 1$ subspace of V with normal e_H.

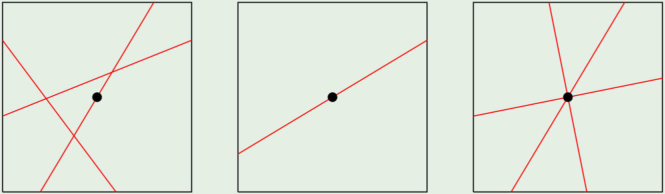
Example



History and Background - Arrangements

- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- Central $\mathcal{A}$ is *essential* if $\{0\} = \bigcap \mathcal{A}$.

Example



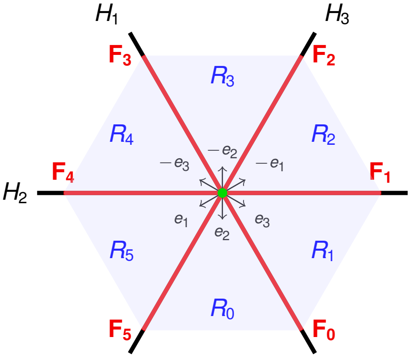
Not central
Not essential

Central
Not essential

Central
Essential

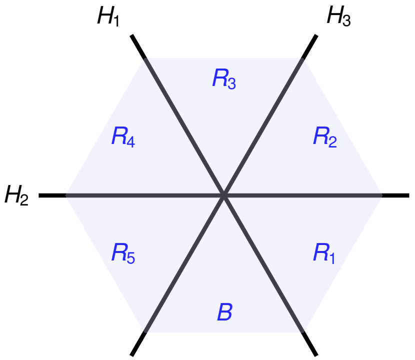
History and Background - Arrangements

- *Regions* $\mathscr{R}_{\mathcal{A}}$ - connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_{\mathcal{A}}$ - intersections of closures of some regions.



History and Background - Poset of regions

- *Base region* $B \in \mathscr{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathscr{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$



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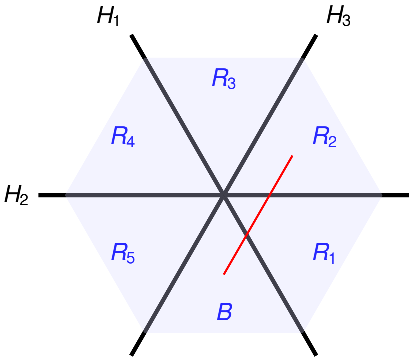

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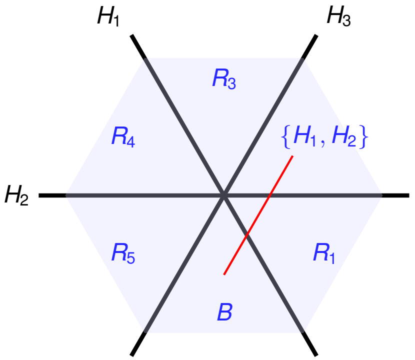

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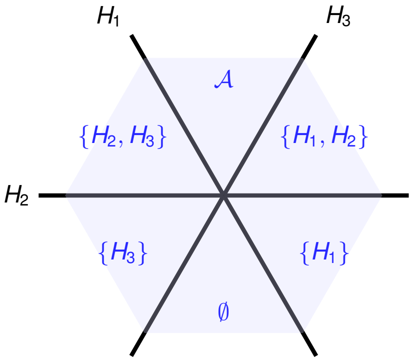
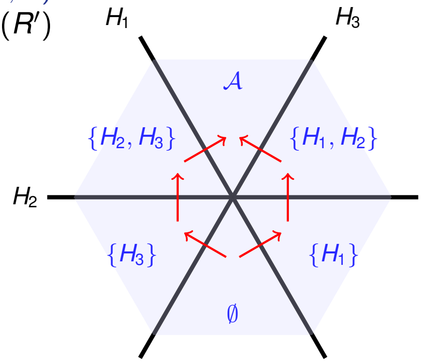


History and Background - Poset of regions

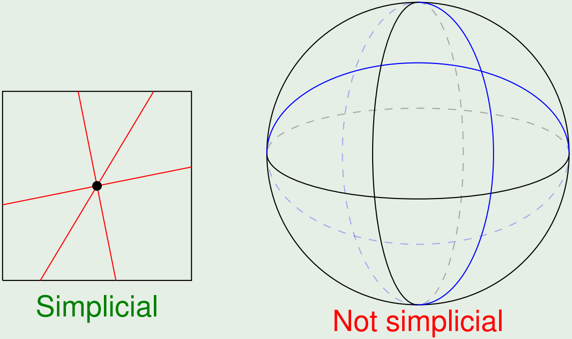
- *Base region* $B \in \mathcal{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathcal{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid \ H \text{ separates } R \text{ from } B\}$
- *Poset of Regions* $\mathrm{PR}(\mathcal{A}, B)$ where
  $R \leq_{\mathrm{PR}} R' \Leftrightarrow S(R) \subseteq S(R')$



History and Background - Poset of regions

- A region R is *simplicial* if normal vectors for boundary hyperplanes are linearly independent.
- $\mathcal{A}$ is *simplicial* if all $\mathscr{R}_{\mathcal{A}}$ simplicial.

Example

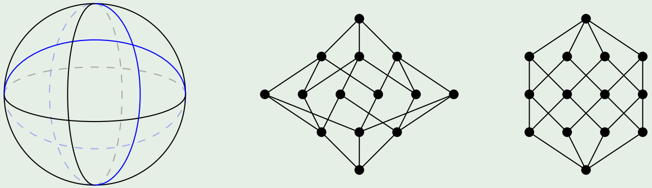


History and Background - Poset of regions

Theorem (Björner, Edelman, Zieglar '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice for any* $B \in \mathscr{R}_{\mathcal{A}}$. *If* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice then B is simplicial.*

Example



History and Background - Poset of regions

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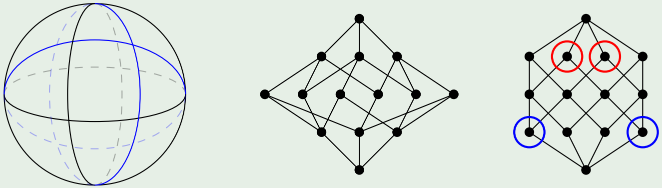

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of Coxeter groups to an order on all the faces of its associated arrangement for type *A* (aka Braid arrangement).
- In 2006, Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.
- In 2016, D, Hohlweg and Pilaud showed this extension has a global equivalent and produces a lattice in Coxeter arrangements.

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- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of Coxeter groups to an order on all the faces of its associated arrangement for type A (aka Braid arrangement).
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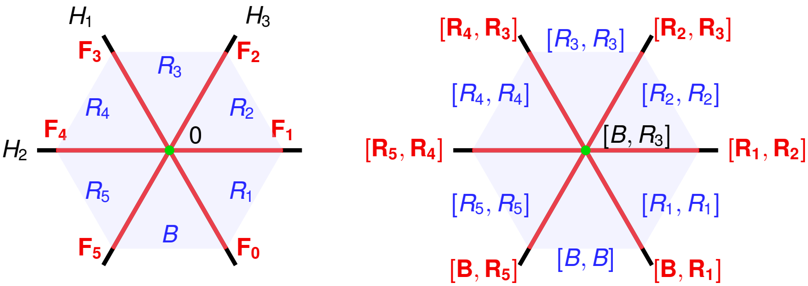
- In 2016, D, Hohlweg and Pilaud showed this extension has a global equivalent and produces a lattice in Coxeter arrangements.
- Questions: Can we extend this to hyperplane arrangements? Can we find both local and global definitions? When do we actually get a lattice?

Facial Intervals

Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_{\mathcal{A}}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that
$$[m_F, M_F] = \left\{ R \in \mathscr{R}_{\mathcal{A}} \mid \ F \subseteq \overline{R} \right\}$$



Facial Weak Order

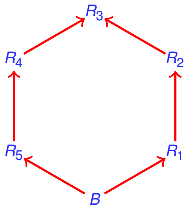
Let $\mathcal{A}$ be a central hyperplane arrangement and B a base region in $\mathscr{R}_{\mathcal{A}}$.

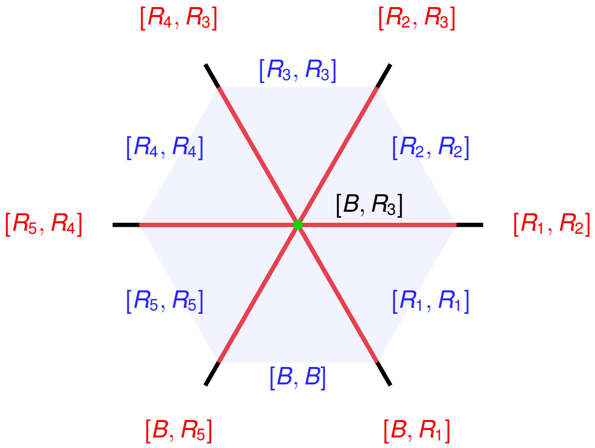
Definition

The *facial weak order* is the order $\mathrm{FW}(\mathcal{A}, B)$ on $\mathscr{F}_{\mathcal{A}}$ where for $F, G \in \mathscr{F}_{\mathcal{A}}$:

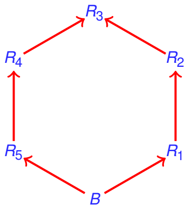
$$F \leq G \Leftrightarrow m_F \leq_{\mathrm{PR}} m_G \text{ and } M_F \leq_{\mathrm{PR}} M_G$$

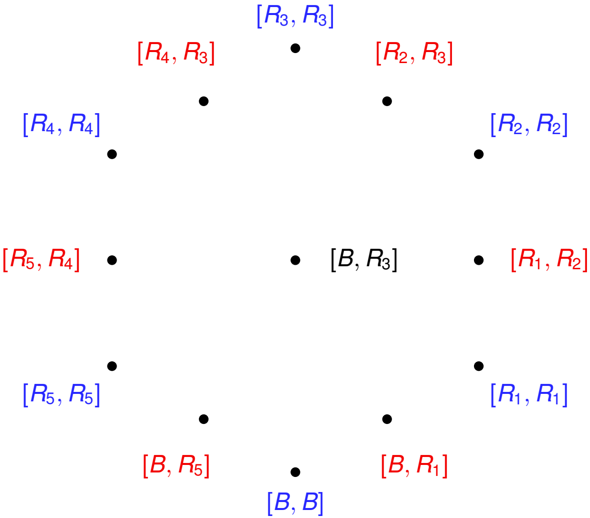
Facial Weak Order - Example



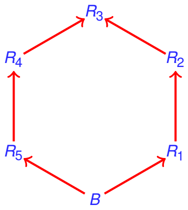


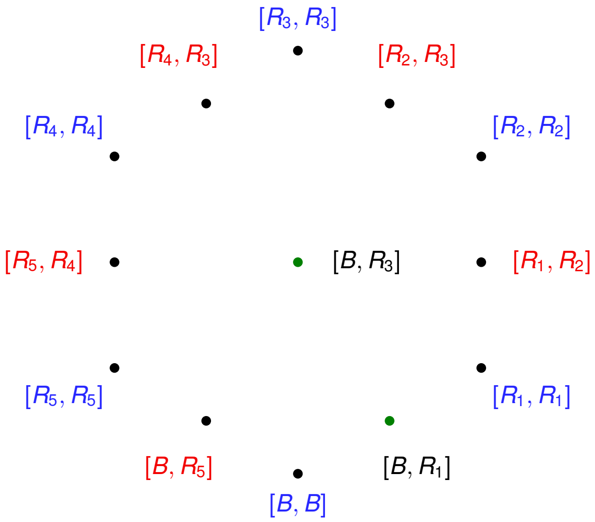
Facial Weak Order - Example





Facial Weak Order - Example





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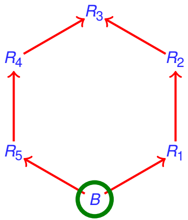

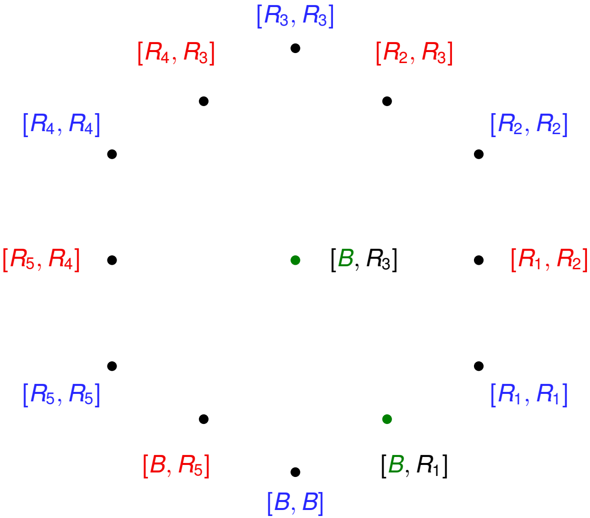

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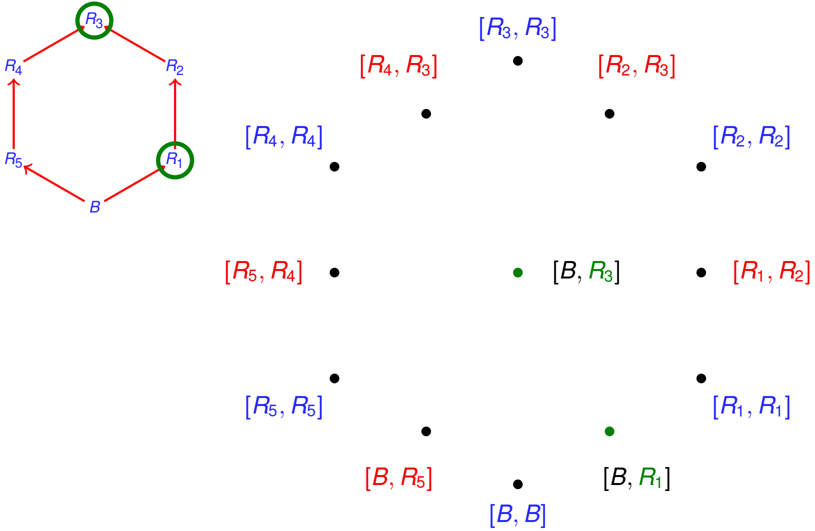

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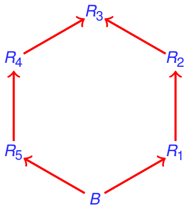

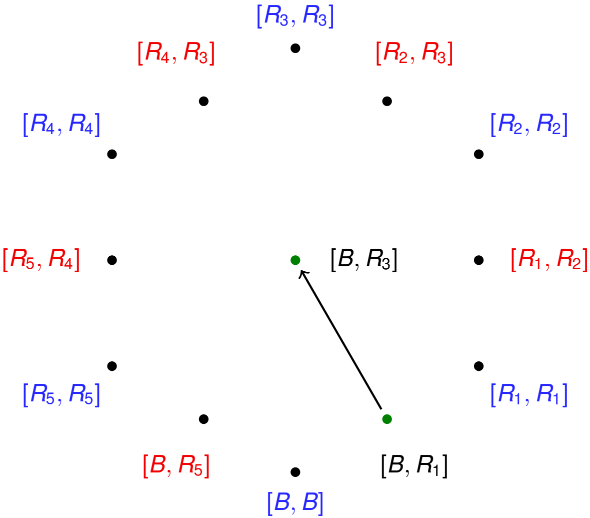

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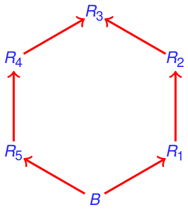
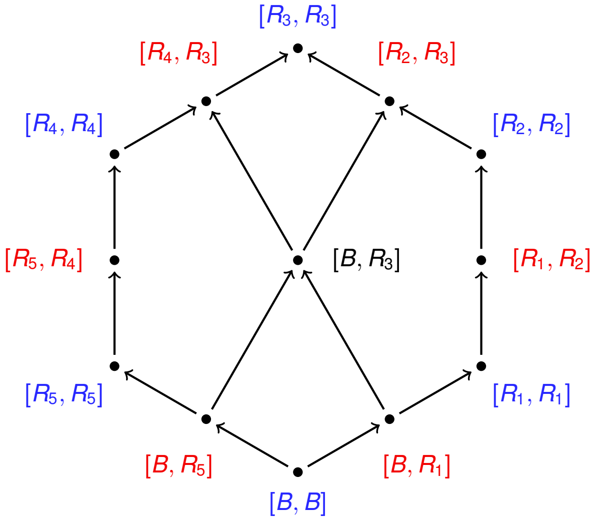




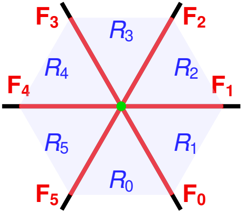
Cover Relations

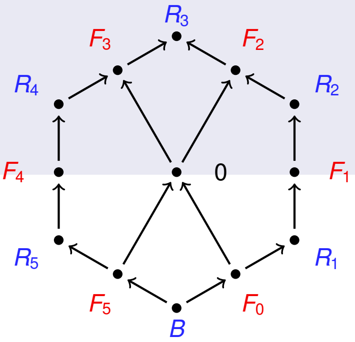
Proposition (D., Hohlweg, McConville, Pilaud, '19+)

For $F, G \in \mathscr{F}_\mathcal{A}$ *if*

1. $F \le G$ *in* $\mathrm{FW}(\mathcal{A}, B)$
2. $|\dim(F) - \dim(G)| = 1$
3. $F \subseteq G$ *or* $G \subseteq F$

then $F \lessdot G$.



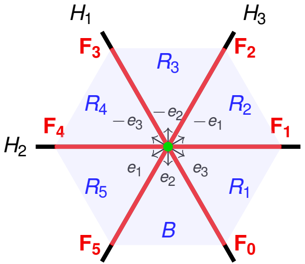


Covectors

- *covector* - a vector in $\{-,0,+\}^{\mathcal{A}}$ with signs relative to hyperplanes.
- $\mathcal{L} \subseteq \{-,0,+\}^{\mathcal{A}}$ - set of covectors

Example

$$F_4 \leftrightarrow (+,0,-) \qquad F_4(H_1) = +;\ F_4(H_2) = 0;\ F_4(H_3) = -$$



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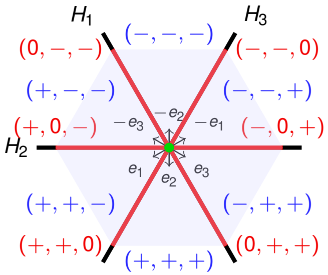
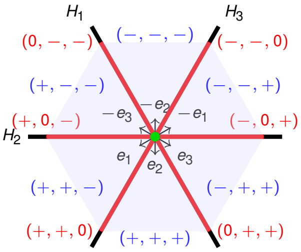


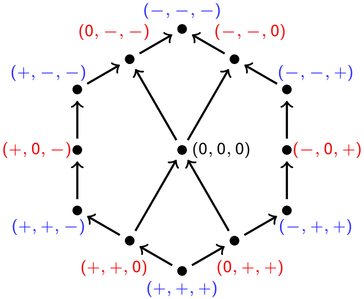
Covector Definition

Definition

For $X, Y \in \mathcal{L}$:

$$X \leq_{\mathcal{L}} Y \Leftrightarrow X(H) \geq Y(H) \quad \forall H \text{ with } - < 0 < +$$





Zonotopes

- *Zonotope* $Z_{\mathcal{A}}$ is the convex polytope:

$$Z_{\mathcal{A}} := \left\{ v \in V \mid \; v = \sum_{i=1}^{k} \lambda_i e_i, \text{ such that } |\lambda_i| \leq 1 \text{ for all } i \right\}$$

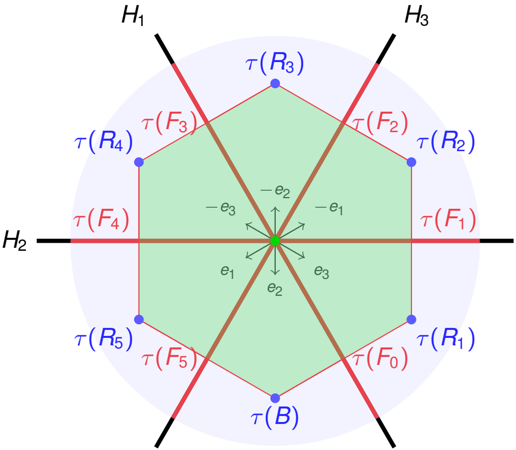
Theorem (Edelman '84, McMullen '71)

There is a bijection between $\mathscr{F}_{\mathcal{A}}$ and the nonempty faces of $Z_{\mathcal{A}}$ given by the map

$$\tau(F) = \left\{ v \in V \mid \; v = \sum_{F(H_i)=0} \lambda_i e_i + \sum_{F(H_j)\neq 0} \mu_j e_j \right\}$$

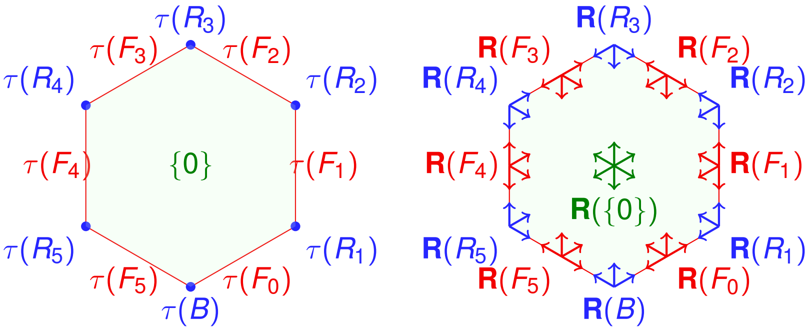
where $|\lambda_i| \leq 1$ for all i and $\mu_j = F(H_j)$

Zonotope - Construction example



Root inversion sets

- *roots* $\Phi_{\mathcal{A}} := \{\pm e_1, \pm e_2, \ldots, \pm e_k\}$
- *root inversion set*
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0 \text{ for some } x \in F\}$.



Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud '19+)

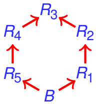
For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

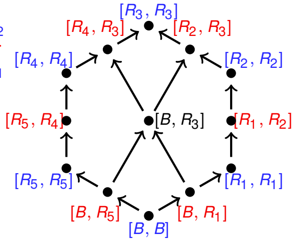
- $m_F \leq_{\mathrm{PR}} m_G$ *and* $M_F \leq_{\mathrm{PR}} M_G$ *in poset of regions* $\mathrm{PR}(\mathcal{A}, B)$.
- *There exists a chain of covers in* $\mathrm{FW}(\mathcal{A}, B)$ *such that*

$$F = F_1 \lessdot F_2 \lessdot \cdots \lessdot F_n = G$$

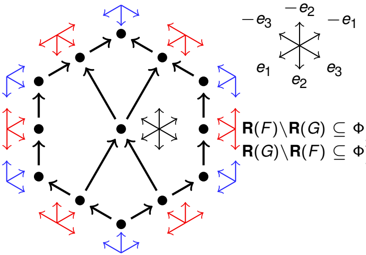
- $F \leq_{\mathcal{L}} G$ *in terms of covectors* $(F(H) \geq G(H)\ \forall H \in \mathcal{A})$
- $\mathbf{R}(F) \backslash \mathbf{R}(G) \subseteq \Phi^{-}_{\mathcal{A}}$ *and* $\mathbf{R}(G) \backslash \mathbf{R}(F) \subseteq \Phi^{+}_{\mathcal{A}}$.

Equivalence for type A_2 Coxeter arrangement



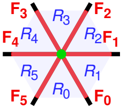


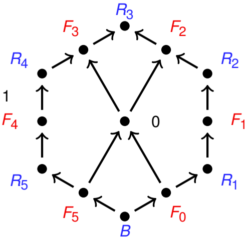
$m_F \leq_{\mathrm{PR}} m_G$
$M_F \leq_{\mathrm{PR}} M_G$

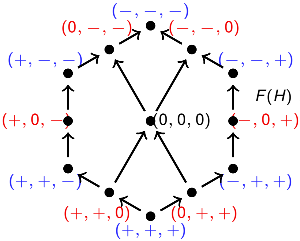


$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$
$\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$

$F \leq G$
$F \subseteq G$ or $G \subseteq F$
$|\dim F - \dim G| = 1$







$F(H) \geq G(H) \;\forall H \in \mathcal{A}$

Facial weak order lattice

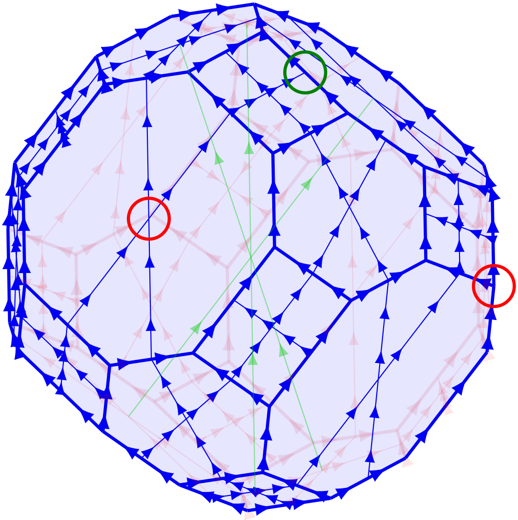
Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice when* $\mathcal{A}$ *is simplicial.*

Corollary (D., Hohlweg, McConville, Pilaud '19+)

The lattice of regions is a sublattice of the facial weak order lattice when $\mathcal{A}$ *is simplicial.*

Example: B_3 Coxeter arrangement



Properties of the facial weak order

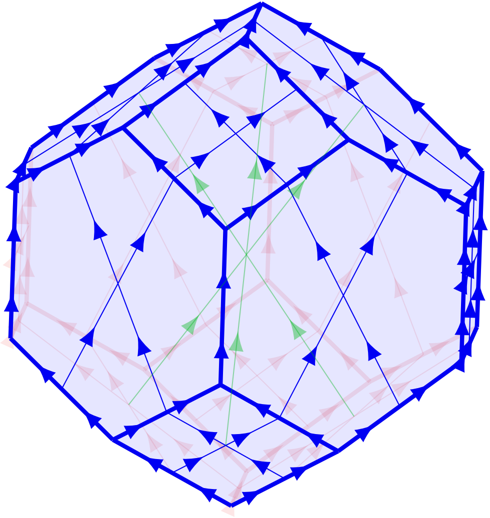
Theorem (D., Hohlweg, McConville, Pilaud '19+)

- $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual.*
- $\mathcal{A}$ *simplicial implies* $\mathrm{FW}(\mathcal{A}, B)$ *is semi-distributive.*
- $\mathcal{A}$ *simplicial and* $X \in \mathscr{F}_{\mathcal{A}}$ *then* X *is join-irreducible in* $\mathrm{FW}(\mathcal{A}, B)$ *if and only if* M_X *is join-irreducible in* $\mathrm{PR}(\mathcal{A}, B)$ *and* $\mathrm{codim}(X) \in \{0, 1\}$
- *Möbius function:* $X, Y \in \mathscr{F}_{\mathcal{A}}$ *let* $Z = X \cap Y$.

$$\mu(X, Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \text{if } X \leq Z \leq Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise} \end{cases}$$

Further Works

- Can we explicitly state the join/meet of two elements?
- When is the facial weak order congruence uniform?
- How many maximal chains are there?
- What is the order dimension?
- Can we generalize this to polytopes?



Thank you!