

Sign Variations and Descents

Aram Dermenjian

Joint with: Nantel Bergeron and John Machacek

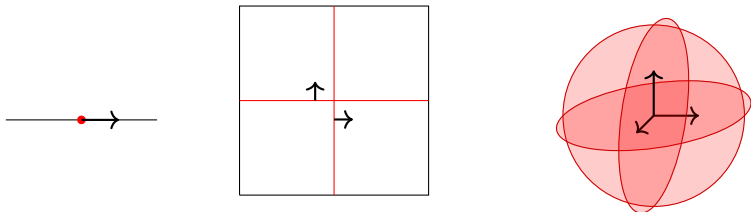
York University

04 December 2020

Sign vectors

- A *sign vector* is a vector in $\mathcal{V}_n = \{+, 0, -\}^n$.
- Think of these as vectors associated to faces in \mathbb{R}^n .

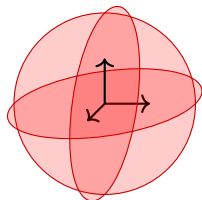
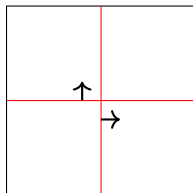
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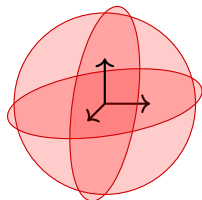
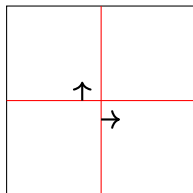
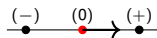
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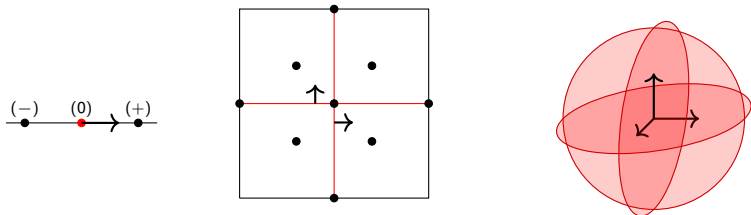
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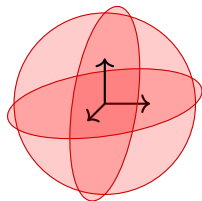
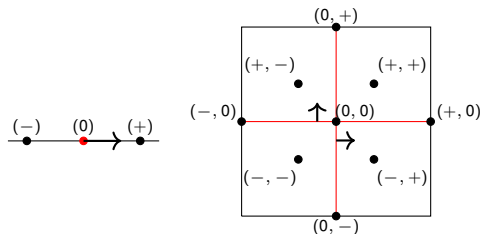
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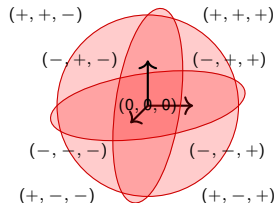
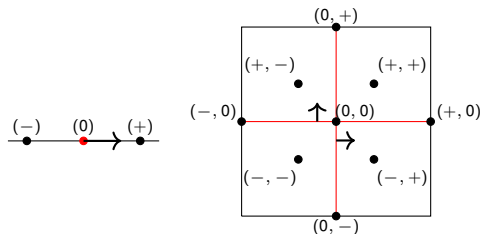
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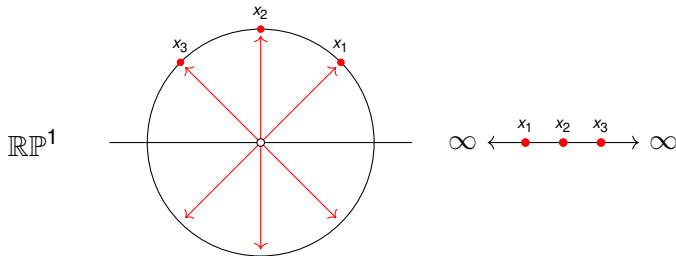
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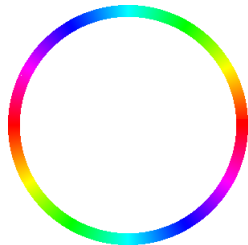


Real Projective Space

- *Real Projective space* \mathbb{RP}^n is quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ under equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R}$.



Real Projective Space - \mathbb{RP}^1



Images from: math.stackexchange.com by Zev Chonoles

Projective sign vectors

- Let \mathcal{PV}_n be the set of sign vectors in \mathbb{RP}^{n-1} .
- In other words, for $\omega \in \mathcal{V}_n$, then $\omega \sim \omega'$ iff $\omega = \omega'$ or $\omega = -\omega'$ and

$$\mathcal{PV}_n = (\mathcal{V}_n \setminus \{0\})^n / \sim .$$

Example

$$\mathcal{V}_1 = \{(+), (0), (-)\} \quad \mathcal{V}_2 = \{(+, +), (+, 0), (+, -), \\ (0, +), (0, 0), (0, -), \\ (-, +), (-, 0), (-, -)\}$$

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Ordering projective sign vectors

- Let P_n denote the poset $(\mathcal{PV}_n, <)$ where for $\omega, \omega' \in \mathcal{PV}_n$:

$$\omega' < \omega \iff \pm\omega' \subseteq \omega$$

in other words, if either ω' or $-\omega'$ is obtained from ω by replacing some components with 0.

Example

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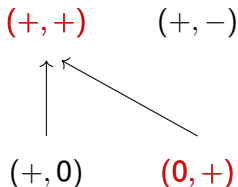
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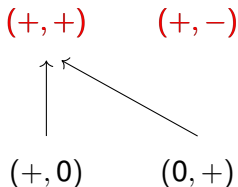
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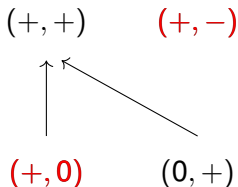
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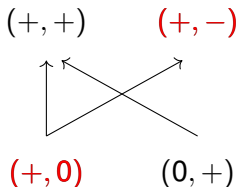
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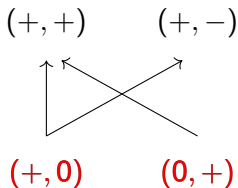
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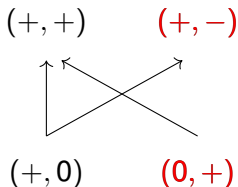
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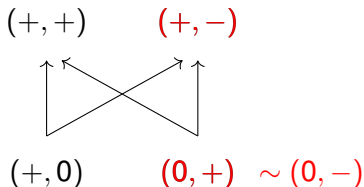
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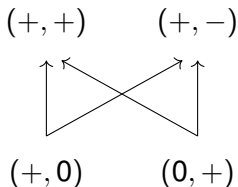
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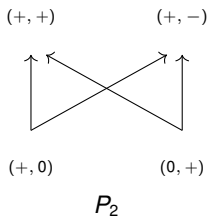
Example



Order complex (of a poset)

- *Simplicial complex* Δ - A collection of sets s.t. $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$.
- The sets are called *faces*. Maximal sets are called *facets*.
- *Order complex* $\Delta(P)$ of a poset P - Simplicial complex where faces are chains in P .

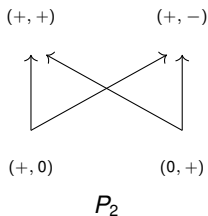
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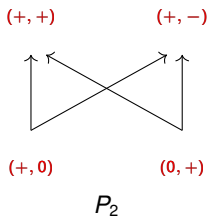


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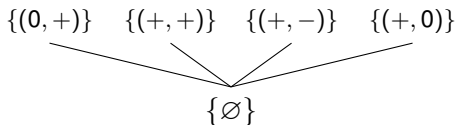
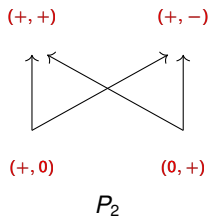


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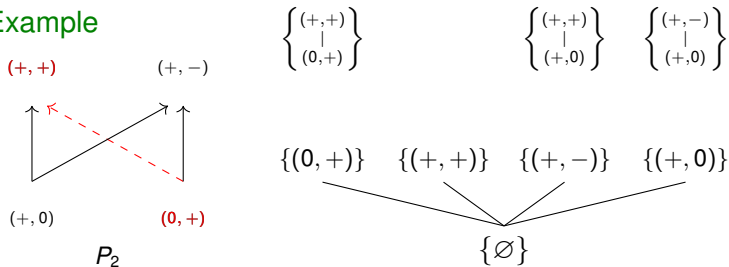
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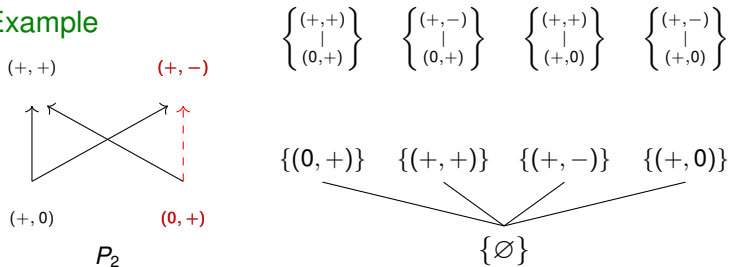
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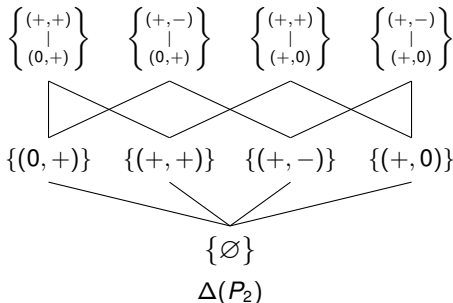
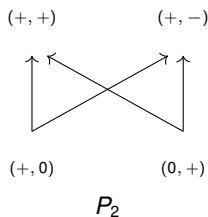
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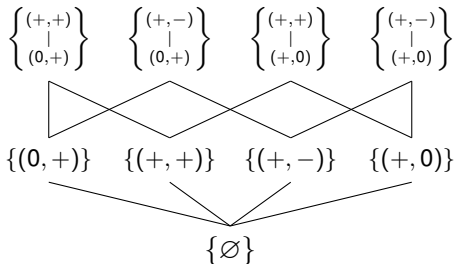
Example



f -vector

- Δ a d -dim simplicial complex.
- f_i = number of i -dim faces
- f -vector is vector faces: $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_d)$.
- $f(\Delta(P))$ is number of elements in each row.

Example



$$f(\Delta(P_2)) = (1, 4, 4)$$

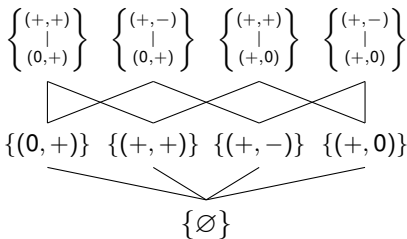
h -vectors

- Δ a d -dim simplicial complex with $f(\Delta) = (f_{-1}, f_0, \dots, f_d)$.

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

- h -vector is vector of h_k s: $h(\Delta) = (h_0, h_1, \dots, h_{d+1})$.

Example



$$f(\Delta(P_2)) = (1, 4, 4)$$

$$h(\Delta(P_2)) = (1, 2, 1)$$

How can we find the h -vector?

Theorem (Stanley 1992(?))

If a simplicial complex Δ is Cohen-Macaulay, its h -vector has nonnegative entries.

Theorem (Machacek 2019)

The order complex $\Delta(P_n)$ is Cohen-Macaulay.

Questions

- *Is there a nice way to compute the h -vector of $\Delta(P_n)$?*

Partitionable simplicial complex

Conjecture (Stanley 1979, Garsia 1980; Counterexample Duval, Goeckner, Klivans, Martin 2016)

Every Cohen-Macaulay simplicial complex is partitionable.

Proposition (Stanley)

If Δ is partitionable, then the partitioning gives us the h -vector.

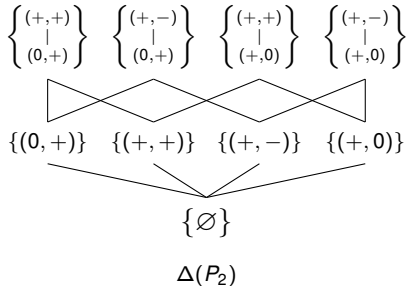
Partitionable

A simplicial complex Δ is *partitionable* if

$$\Delta = \bigsqcup [G_i, F_i]$$

where F_i is a facet.

Example



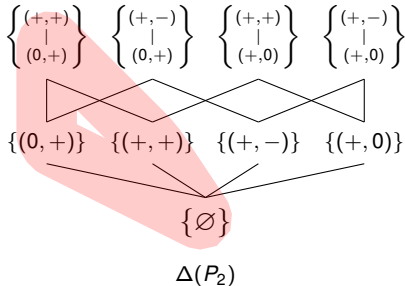
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Example



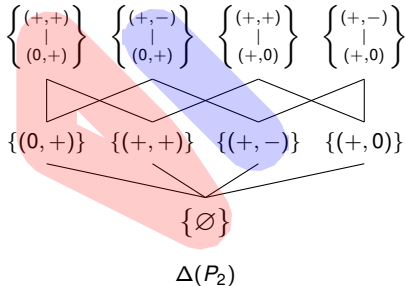
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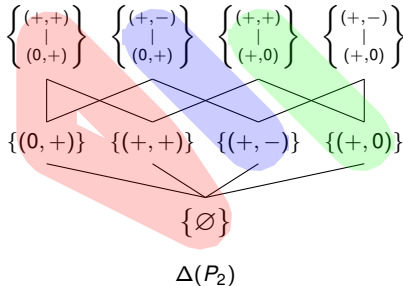
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A simplicial complex Δ is *partitionable* if

$$\Delta = \bigsqcup [G_i, F_i]$$

where F_i is a facet.

Example



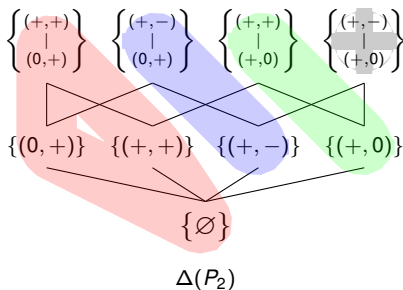
Partitionable

A simplicial complex Δ is *partitionable* if

$$\Delta = \bigsqcup [G_i, F_i]$$

where F_i is a facet.

Example



Partitionable

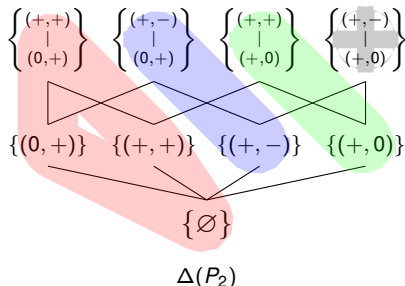
Proposition (Stanley)

Let Δ be a partitionable simplicial complex with partitioning $\Delta = \sqcup [G_i, F_i]$ where F_i is a facet. Then

$$h_i(\Delta) = |\{j : |G_j| = i\}|.$$

Example

$$h(\Delta(P_2)) = (h_0, h_1, h_2)$$



Partitionable

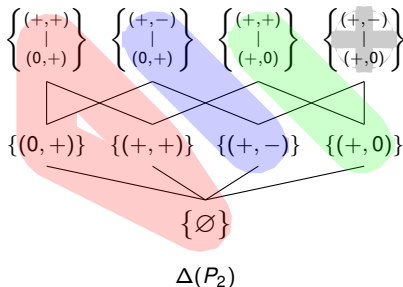
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Let Δ be a partitionable simplicial complex with partitioning $\Delta = \sqcup [G_i, F_i]$ where F_i is a facet. Then

$$h_i(\Delta) = |\{j : |G_j| = i\}|.$$

Example

$$h(\Delta(P_2)) = (1, h_1, h_2)$$



Partitionable

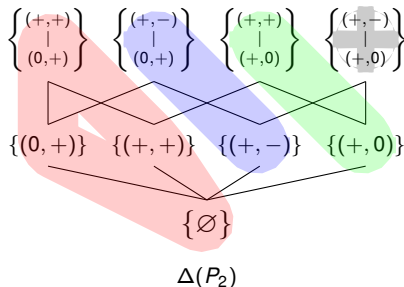
Proposition (Stanley)

Let Δ be a partitionable simplicial complex with partitioning $\Delta = \sqcup [G_i, F_i]$ where F_i is a facet. Then

$$h_i(\Delta) = |\{j : |G_j| = i\}|.$$

Example

$$h(\Delta(P_2)) = (1, 2, 1)$$



Main Theorem

Let D_n be a type D Coxeter group and let des_B denote the type B descent set of an element $\pi \in D_n$.

Theorem (Bergeron, D., Machacek 2020BP)

The order complex $\Delta(P_n)$ is partitionable. Moreover,

$$h_i(\Delta(P_n)) = |\{\pi \in D_n : |\text{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Coxeter groups

Type A_n

The elements in type A_n Coxeter groups can be represented as permutations in \mathfrak{S}_{n+1} .

$$57238146 \in A_7$$

Type B_n

The elements in type B_n Coxeter groups can be represented as *signed* permutations of \mathfrak{S}_n .

$$5\bar{7}23\bar{8}\bar{1}46 \in B_8$$

Type D_n

The elements in type D_n Coxeter groups can be represented as *even signed* permutations of \mathfrak{S}_n .

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12345678$$

$$\text{des}_A(\pi) = \emptyset$$

Descents - Type A

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$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12345678$$

$$\text{des}_A(\pi) = \{2\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57\mathbf{2}38146 \in A_7$$

$$12\mathbf{3}45678$$

$$\text{des}_A(\pi) = \{2\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 572\mathbf{38}146 \in A_7$$

$$123\mathbf{45}678$$

$$\text{des}_A(\pi) = \{2\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12345678$$

$$\text{des}_A(\pi) = \{2\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12345678$$

$$\text{des}_A(\pi) = \{2, 5\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12345678$$

$$\text{des}_A(\pi) = \{2, 5\}$$

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For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$12346578$$

$$\text{des}_A(\pi) = \{2, 5\}$$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\text{des}_A(\pi)$ denote the descent set of π .

$$\text{des}_A(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

$$\pi = 57238146 \in A_7$$

$$\text{des}_A(\pi) = \{2, 5\}$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

To find descent, we add a 0 in front, and calculate like “normal”.

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\text{des}_B(\pi) = \emptyset$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$01\bar{2}345678$$

$$\text{des}_B(\pi) = \{1\}$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012\bar{3}45678$$

$$\text{des}_B(\pi) = \{1, 3\}$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\text{des}_B(\pi) = \{1, 3, 4\}$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\text{des}_B(\pi)$ denote the descent set of π .

$$\text{des}_B(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$\text{des}_B(\pi) = \{1, 3, 4\}$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\text{des}_D(\pi)$ denote the descent set of π .

$$\text{des}_D(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_1$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\text{des}_D(\pi)$ denote the descent set of π .

$$\text{des}_D(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

To find descent, we add a 7 in front, and calculate like “normal”.

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\text{des}_D(\pi) = \emptyset$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\text{des}_D(\pi)$ denote the descent set of π .

$$\text{des}_D(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\text{des}_D(\pi) = \{0\}$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\text{des}_D(\pi)$ denote the descent set of π .

$$\text{des}_D(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$7\bar{5}\bar{2}\bar{3}\bar{8}\bar{1}46$$

$$0\bar{1}\bar{2}345678$$

$$\text{des}_D(\pi) = \{0, 1\}$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\text{des}_D(\pi)$ denote the descent set of π .

$$\text{des}_D(\pi) = \{i : \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012\bar{3}45678$$

$$\text{des}_D(\pi) = \{0, 1, 3\}$$

Descents - Type D

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Example

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$$\text{des}_D(\pi) = \{0, 1, 3, 4\}$$

Descents - Type D

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Example

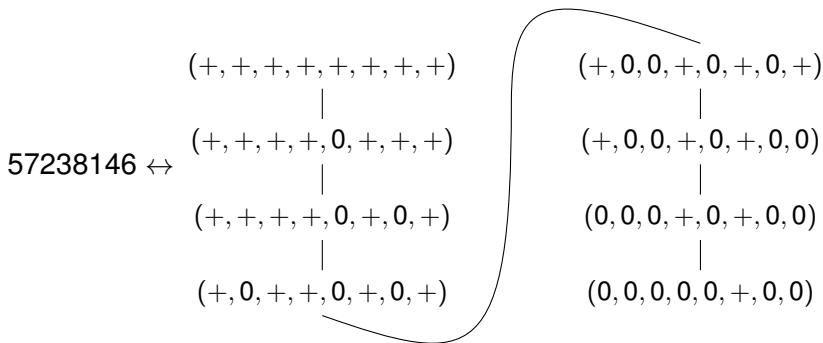
$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$\text{des}_D(\pi) = \{0, 1, 3, 4\}$$

Permutations and maximal chains

How do we associate permutations and maximal chains in our poset?

- Change π_j to 0 inductively.



Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

57238146

Permutations and chains

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57|238|146 $\xrightarrow{\text{min}}$

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$$57|238|146 \xrightarrow{\min} (+, +, +, +, 0, +, 0, +)$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$57|238|146 \xrightarrow{\min} \begin{array}{c} (+, +, +, +, 0, +, 0, +) \\ | \\ (+, 0, 0, +, 0, +, 0, 0) \end{array}$$

Permutations and chains

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$$57|238|146 \xrightarrow{\text{min}} \begin{array}{c} (+, +, +, +, 0, +, 0, +) \\ | \\ (+, 0, 0, +, 0, +, 0, 0) \end{array}$$

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$$57238146 \xrightarrow{\min} \begin{array}{c} (+, +, +, +, 0, +, 0, +) \\ | \\ (+, 0, 0, +, 0, +, 0, 0) \end{array}$$

$$\text{des}_A(\pi) = \{2, 5\}$$

Permutations and chains

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- Order 0s first
- Add increasing sequences inductively.

$$\begin{array}{ccc} & (+, +, +, +, 0, +, 0, +) & \\ & | & \\ 57238146 \xrightarrow{\min} & (+, 0, 0, +, 0, +, 0, 0) & \rightarrow \\ & \text{des}_A(\pi) = \{2, 5\} & \end{array}$$

Permutations and chains

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$$\begin{array}{ccc} & (+, +, +, +, 0, +, 0, +) & \\ & | & \\ 57238146 \xrightarrow{\min} & (+, 0, 0, +, 0, +, 0, 0) & \rightarrow 57 \\ & \text{des}_A(\pi) = \{2, 5\} & \end{array}$$

Permutations and chains

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- Order 0s first
- Add increasing sequences inductively.

$$57238146 \xrightarrow{\min} \begin{array}{c} (+, +, +, +, 0, +, 0, +) \\ | \\ (+, 0, 0, +, 0, +, 0, 0) \end{array} \rightarrow 57238$$

$$\text{des}_A(\pi) = \{2, 5\}$$

Permutations and chains

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$$\begin{array}{ccc} & (+, +, +, +, 0, +, 0, +) & \\ & | & \\ 57238146 \xrightarrow{\min} & (+, 0, 0, +, 0, +, 0, 0) & \rightarrow 57238146 \\ & \text{des}_A(\pi) = \{2, 5\} & \end{array}$$

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Negatives?

But how do we handle the negatives?!

5 $\bar{7}$ 2 $\bar{3}$ 8 $\bar{1}$ 46

\leftrightarrow

?

Negatives?

But how do we handle the negatives?!

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \quad \leftrightarrow \quad ?$$

$(57238146, \{1, 3, 7, 8\})$

Sign Variations

Sign vector $\omega \in \mathcal{V}_n = \{+, 0, -\}^n$.

$\text{var}(\omega)$ = number of times ω changes sign

$i \in [n]$ is a *sign flip* of ω if there exists a j such that $\omega_{i-j}\omega_i < 0$ while $\omega_{i-k}\omega_i = 0$ for all $1 \leq k < j$.

Example

$$\omega = (+, +, -, -, -, -, +, -) \Rightarrow \text{var}(\omega) = 3$$

$$\begin{array}{cccccccc} & \frown & & \frown & \frown & \frown & & \\ (+, +, -, -, -, -, +, -) & \leftrightarrow & \{3, 7, 8\} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

Cyclic Sign Variations

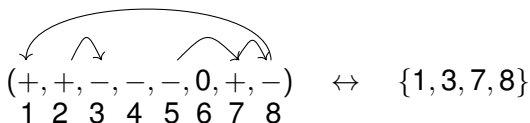
Sign vector $\omega \in \mathcal{V}_n = \{+, 0, -\}^n$.

$\text{cvar}(\omega)$ = number of times ω changes sign, cyclically

$i \in [n]$ is a *cyclic sign flip* of ω if there exists a j such that $\omega_{i-j}\omega_i < 0$ while $\omega_{i-k}\omega_i = 0$ for all $1 \leq k < j$ where $\omega_i = \omega_{i+n}$.

Example

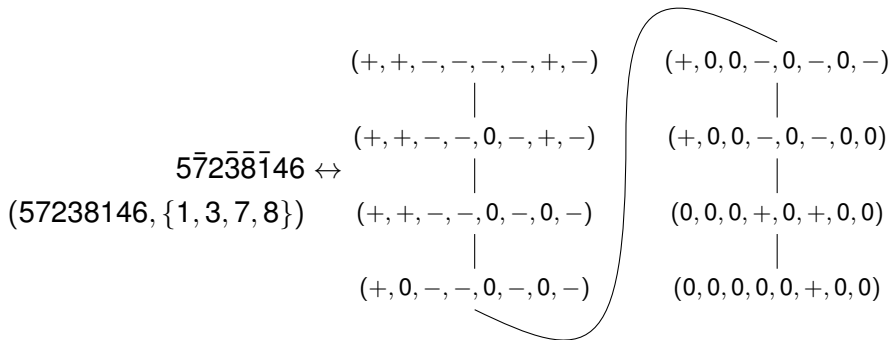
$$\omega = (+, +, -, -, -, -, +, -) \Rightarrow \text{cvar}(\omega) = 4$$



Even signed permutations and maximal chains

How about even signed permutations and maximal chains?

- Just as before with a final step of adding negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46$$
$$(57238146, \{1, 3, 7, 8\})$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} (57238146, \{1, 3, 7, 8\})$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

(+, +, -, -, -, -, +, -)

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\min} (57238146, \{1, 3, 7, 8\})$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$(+, +, -, -, -, -, +, -)$

$(+, +, -, -, 0, -, +, -)$

$5\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}}$

$(57238146, \{1, 3, 7, 8\})$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

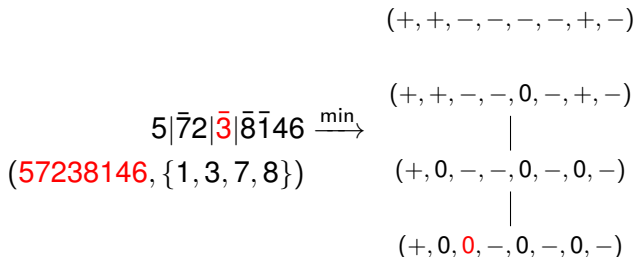
- Just as before, but with negatives.

$$\begin{array}{ccc}
 & & (+, +, -, -, -, -, +, -) \\
 & & \\
 & & (+, +, -, -, 0, -, +, -) \\
 5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} & & | \\
 (57238146, \{1, 3, 7, 8\}) & & (+, 0, -, -, 0, -, 0, -)
 \end{array}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

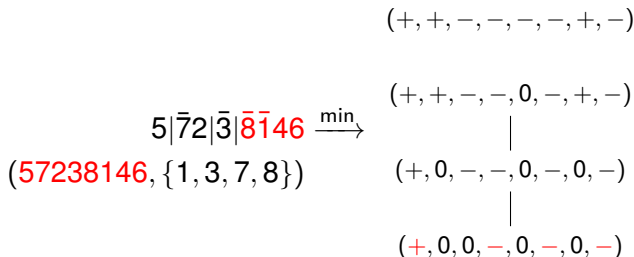
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$\begin{array}{r}
 5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} \\
 (57238146, \{1, 3, 7, 8\})
 \end{array}
 \begin{array}{l}
 (+, +, -, -, -, -, +, -) \\
 (+, +, -, -, 0, -, +, -) \\
 | \\
 (+, 0, -, -, 0, -, 0, -) \\
 | \\
 (+, 0, 0, -, 0, -, 0, -) \\
 \text{des}_D(\pi) = \{0, 1, 3, 4\}
 \end{array}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$\begin{array}{r}
 5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} \\
 (57238146, \{1, 3, 7, 8\})
 \end{array}
 \begin{array}{l}
 (+, +, -, -, -, -, +, -) \\
 (+, +, -, -, 0, -, +, -) \\
 | \\
 (+, 0, -, -, 0, -, 0, -) \\
 | \\
 (+, 0, 0, -, 0, -, 0, -) \\
 \text{des}_D(\pi) = \{0, 1, 3, 4\} \\
 \text{des}_B(\pi) = \{1, 3, 4\}
 \end{array}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

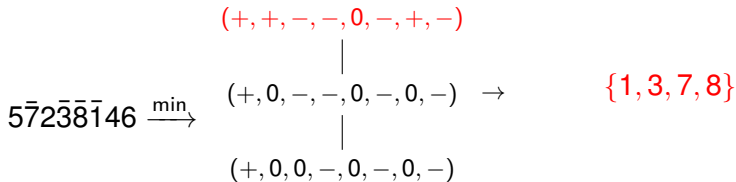
- Just as before, but with negatives.

$$\begin{array}{ccc}
 & & (+, +, -, -, 0, -, +, -) \\
 & & | \\
 5\bar{7}2\bar{3}\bar{8}\bar{1}4\bar{6} \xrightarrow{\min} & & (+, 0, -, -, 0, -, 0, -) \rightarrow \\
 & & | \\
 & & (+, 0, 0, -, 0, -, 0, -)
 \end{array}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

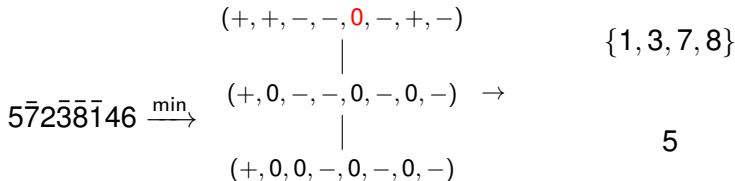
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

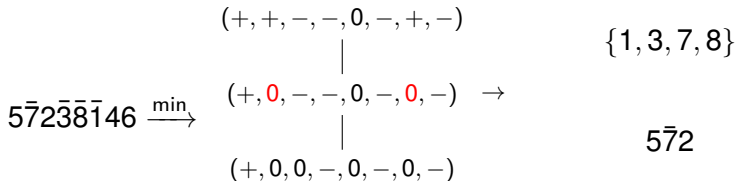
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

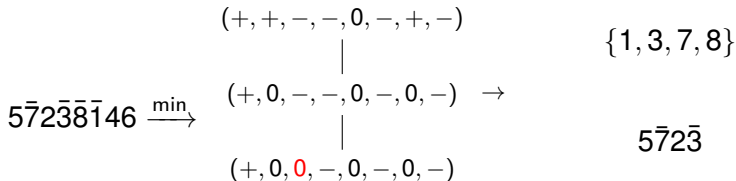
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

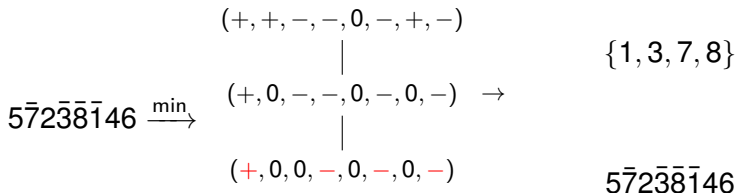
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

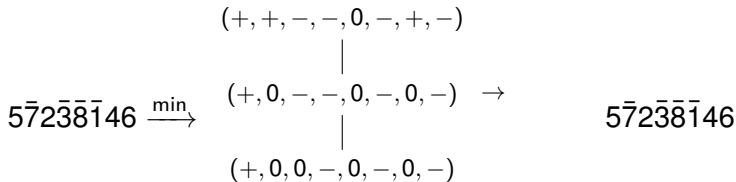
- Just as before, but with negatives.



Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.



Main Theorem (Again)

Let D_n be the type D Coxeter group and let des_B denote the type B descent set of an element $\pi \in D_n$.

Theorem (Bergeron, D., Machacek 2020BP)

The order complex $\Delta(P_n)$ is partitionable. Moreover,

$$h_i(\Delta(P_n)) = |\{\pi \in D_n : |\text{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Restriction of variations

- $\mathcal{PV}_{n,m} = \{\omega \in \mathcal{PV}_n : \text{var}(\omega) \leq m\}$.
- $P_{n,m} = (\mathcal{PV}_{n,m}, <)$.
- $D_{n,m} = \{\pi \in D_n : \pi \text{ has at most } m \text{ negatives}\}$.

Theorem (Bergeron, D., Machacek 2020BP)

If $m \leq n - 1$ is even then the order complex $\Delta(P_{n,m})$ is partitionable. Moreover,

$$h_i(\Delta(P_{n,m})) = |\{\pi \in D_{n,m} : |\text{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Thank you!

$$\Delta(P_3)$$

